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Multiplicative function :- An arithmetic function $f$ is said to be multiplicative if

$$f(mn) = f(m) f(n)$$

where $(m, n) = 1$.

Th 1 :- The arithmetic functions $\tau$ and $\sigma$ are multiplicative.

Proof )- For $m = n = 1$ the result is true.
Let $m > 1$ and $n > 1$ be two relatively prime integers i.e. $(m, n) = 1$. Let $m = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ and $n = q_1^{b_1} q_2^{b_2} \cdots q_s^{b_s}$ be prime factorization of $m$ and $n$. Since $(m, n) = 1$ no $p_i$ will be among $q_j$ and vice-versa.
Now prime factorization of $mn$ will be

$$mn = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} q_1^{b_1} q_2^{b_2} \cdots q_s^{b_s}$$

Hence

$$\tau(mn) = [(a_1+1)(a_2+1) \cdots (a_r+1)] \cdot [(b_1+1)(b_2+1) \cdots (b_s+1)] \quad \text{— (1)}$$

$$= \tau(m) \tau(n)$$

Similarly

$$\sigma(mn) = \sigma(m) \sigma(n)$$

prove urself

Ex :- Evaluate $\tau$ and $\sigma$ for $n = 3000$

Proof :- we have

$$3000 = 2^3 \, 3^1 . 5^3$$

$$\tau(3000) = (3+1).(1+1).(3+1)$$

$$= 4 . 2 . 4$$

$$= 32$$

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$$\sigma(3000) = \frac{(2^4-1)}{2-1}\,\frac{(3^2-1)}{3-1}\,\frac{(5^4-1)}{5-1}$$

$$= \frac{(16-1)}{1}\,\frac{(9-1)}{2}\,\frac{(625-1)}{4}$$

$$= 15.4.156 = 9360$$

5 Behaviour of $\tau(n)$ as $n \to \infty$ :- For every prime $p$ we have $\tau(p) = 2$. For any integer $n$ there are at least two divisors 1 and $n$. thus $\tau(n) \geq 2$ for all $n$. For any intega of the form $n = p^a$, $p$ is a prime we hae $\tau(n) = a+1$. This depends on a. for large a, $\tau(n)$ is also sufficiently large. Thus $\lim_{n\to\infty} \tau(n) = \infty$. The behaa of $\tau(n)$ is irregular.

The Mobius function :- For a positive integer $n$ the Mobius function $\mu$ is defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } p^2/n \text{ for some prime } p \\ (-1)^k & \text{if } n = p_1 p_2 \cdots p_k \end{cases}$$

For example

$$\mu(6) = \mu(2.3) = (-1)^2 = 1$$

$$\mu(9) = \mu(3^2) = 0$$